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Distortion of turbulence in flows with parallel streamlines

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Velocity moments have been calculated for the rapid distortion of axisymmetric turbulence in a uniform mean shear. The moments are compared with data on steady pipe flows and two-dimensional channel flows, and the turbulence structure of these flows is summarized in terms of effective distortion strain. The centre-line structure is taken to be characteristic of the undistorted state which from the data is more nearly axisymmetric rather than isotropic. A closer comparison is found than that by Townsend (1970), and in particular the differences in stress ratios $\tau/\rho\overline{u_1^2}$, $\tau/\rho\overline{q^2}$ found between different experiments can be accounted for with the hypothesis of initially axisymmetric turbulence. Profiles for the effective strain are derived from the experiments and are shown to have the same form and to indicate the existence of a relaxation timescale for the large eddies, comparable to the energy decay timescale. An equation for the effective distortion strain is formulated that can be incorporated into a turbulence model.

1. Introduction

Over the last decade there has been an intensified interest in developing models for turbulent shear flows based on moment-transport equations. These models are more sophisticated than earlier methods such as the use of an eddy viscosity and provide greater flexibility and accuracy. The basic approach as set out by Launder, Reece & Rodi (1975) is to assume that the turbulence is locally homogeneous and that the various terms in the moment equations may be represented by tensorial relations in the instantaneous, local values for the velocity moments $\overline{u_i u_j}$ and the turbulence dissipation ϵ. The methods for prescribing these tensorial relations are reviewed by Lumley (1978). This approach has been quite successful in estimating equilibrium flows where the turbulence has sufficient time compared with changes in the mean flow to maintain an asymptotic equilibrium structure. However, in rapidly accelerating flows or unsteady shear flows it is not obvious that the assumption of a local time dependence is still suitable, and it may be necessary to consider the history of the turbulence structure.

An alternative approach that does not assume a local time dependence is to use the results of rapid-distortion theory for turbulence in a uniform shear flow. The formal assumption of rapid distortion is that the distortion by the mean shear is much stronger than the fluctuating strain rates of the larger eddies. In particular, the inertial transfer of energy to smaller scales is weak. For rapidly evolving flows where significant distortions occur on a timescale short compared with the timescale for the

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energy decay of the larger eddies, the use of rapid-distortion theory can be formally justified for short time intervals.

Townsend (1970, 1976) has further shown that the structure of turbulence in homogeneous shear flows and in simple shear flows such as wakes, channel flows, and boundary layers is also fairly well described by rapid-distortion theory, even though the condition of a rapid distortion is not satisfied. In particular, he has shown that two-point velocity correlations and the ratios of the velocity moments have essentially the correct form and can be characterized by an effective value of the distortion strain parameter. The development of the turbulence structure at a particular point in the flow may then be associated with a particular distortion strain, and in an equilibrium flow this strain will have some asymptotic value. Townsend (1980) has followed up this approach in studying the behaviour of a distorted wake and the flow in a curved mixing layer. The shear flows in these cases were assumed to have some equilibrium strain before entering the contraction or curved flow region, where they were then subjected to additional distortion. In both cases rapid distortion gave good estimates of the ratio of Reynolds shear stress τ to turbulent kinetic energy $\frac{1}{2}\rho\overline{q^2}$. The curved mixing layer has also been investigated by Gibson & Rodi (1981) using a moment-transport equation model. Their results also compared favourably with the experiments of Castro & Bradshaw (1976) in this example.

As pointed out by Townsend (1980), rapid-distortion theory in itself does not provide a practical turbulence model. The theory is based on the linearization of the velocity-fluctuation equations and the neglect of nonlinear inertial processes of the turbulence, important in particular to the decay of the large eddies. Further, an arbitrary velocity scale is introduced by this linear theory. To formulate a useful model the results of rapid distortion must be limited to specifying stress ratios $\overline{u_i u_j}/\overline{q^2}$, and separate equations must be given to specify the turbulent kinetic energy and the effective distortion strain of the turbulence. A proposal along these lines was made by Mathieu (1971), modifying the model of Bradshaw, Ferriss & Atwell (1967). Bradshaw's method assumed that the stress ratio $\tau/\rho\overline{q^2}$ had a uniform, constant value of typically 0·15, and then used a turbulent-energy equation to determine turbulent kinetic energy and hence shear stress. Mathieu proposed instead that the stress ratio be estimated in terms of the net strain distortion by the mean shear following a mean-flow pathline. The turbulence was assumed to be either initially undistorted, as in entrainment of free-stream turbulence, or else assigned some appropriate initial strain. A transport equation for this stress ratio was formulated and a relation for the stress ratio to distortion strain obtained empirically from the data of Champagne, Harris & Corrsin (1970). The details of this model and its applications are given by Jeandel, Brison & Mathieu (1978).

The purpose of this paper is to re-examine the description of some equilibrium shear flows in terms of the results of rapid-distortion theory and to quantify the effective distortion strain parameter. In §3, data from some channel-flow and pipe-flow experiments will be examined and profiles of effective strain derived. These will then be used to formulate a strain equation linking the two limiting cases of rapid distortion and equilibrium shear structure. The results of rapid distortion are usually derived for initially isotropic turbulence for simplicity and on the basis that the distorted structure will not be sensitive to the initial spectrum. This is not the case in fact, and in making comparisons with shear flows a more general hypothesis is needed. Instead it will be assumed here that the undistorted turbulence is initially axisymmetric and results for the shear distortion of initially axisymmetric turbulence are presented in §2.

2. Rapid distortion of axisymmetric turbulence by a uniform shear

The assumptions and applications for rapid distortion theory in general have been reviewed by Hunt (1978). The standard formulation for shear distortion is to suppose that a field of initially homogeneous and isotropic turbulence is suddenly subjected to a strong uniform mean shearing flow whose mean strain rate is much greater than the typical strain rate of the larger, more energetic eddies. The characteristics of these larger energetic eddies, as opposed to the less energetic dissipative motions, are then determined from considering the effect of the mean shear distortion. The nonlinear inertial processes such as energy transfer to smaller scales are neglected in comparison with the effects of mean shear. This assumption is valid for time intervals shorter than the timescale for energy decay. For a mean shear flow

$$\mathbf{U} = (\beta(t)\, x_3, 0, 0), \tag{2.1}$$

and turbulence characterized by a typical initial r.m.s. velocity scale u_0 and integral lengthscale L_0, these assumptions require that the ratio $u_0/\beta L_0$ be small. With these assumptions the equations for fluctuating momentum may be linearized and the resulting problem solved in terms of Fourier components of the velocity fluctuations, as given by Moffatt (1967) and Townsend (1970). These solutions are then combined to obtain the spectrum tensor and velocity correlations. An alternative approach is to form linearized equations for the two-point velocity correlations and to solve directly for the spectrum tensor, neglecting third-order and higher moments. This latter approach has been adopted by Pearson (1959), Deissler (1961, 1970), Fox (1964), Loiseau (1973) and Courseau (1974). The works of Deissler and of Pearson were based on a low-Reynolds-number assumption neglecting nonlinear processes compared with viscous effects rather than a rapid distortion, thus allowing them to consider long-time-interval limits, although the mathematical problems are equivalent. The approach of Townsend (1970) is followed here, and the results are summarized in the appendix. In addition to the assumption of rapid distortion the effects of viscosity ν are neglected, assuming that the direct effect of viscosity on the more energetic eddies is small compared with the effects of mean distortion, so that the ratio $\beta L_0^2/\nu$ is large.

The rapid-distortion approximation relates the general two-point time velocity correlation $\overline{u_i(\mathbf{x}, t)\, u_j(\mathbf{x}+\mathbf{r}, t+s)}$ to the spectrum tensor $\Phi_{ij}(\mathbf{m})$ of the initial undistorted homogeneous turbulence as

$$\overline{u_i(\mathbf{x}, t)\, u_j(\mathbf{x}+\mathbf{r}, t+s)} = \int d^3\mathbf{m}\, A_{ip}(\mathbf{m}, \alpha(t))\, A_{jq}(\mathbf{m}, \alpha(t+s)) \times \Phi_{pq}(\mathbf{m}) \exp\{i[\mathbf{m}\,.\,\mathbf{r} + m_1(x_3\alpha(t) - x_3\alpha(t+s) - r_3\alpha(t+s))]\}. \tag{2.2}$$

The distortion matrix $\mathbf{A}$ is defined in the appendix, and for no distortion it reduces to the identity matrix. The distortion strain parameter $\alpha(t)$ characterizes the effect of the mean shear on the turbulence structure and is defined by

$$\alpha(t) = \int_0^t \beta(t')\, dt'. \tag{2.3}$$

Once an initial spectrum is specified, correlations and velocity moments may be evaluated from (2.2). The general two-point, two-time correlation is not homogeneous, however, even if the initial turbulence is, as may be seen from (2.2). This is an effect of the differential mean-flow advection. The term $x_3(\alpha(t) - \alpha(t+s))$ is the streamwise displacement produced by the mean flow between the two times t and $t+s$. In a coordinate system moving with the mean shear flow the statistics do remain homogeneous. The single-time correlations for $s = 0$ and the velocity moments $\overline{u_i u_j}(t)$ will remain uniform.

The initial spectrum tensor Φ_{ij} is chosen here to be axisymmetric rather than isotropic, as assumed by most previous authors. The exception to this is the work of Deissler (1975), who sought a closer comparison between rapid shear distortion and the experimental observations of Champagne *et al.* (1970). He allowed for a complicated form of the initial spectrum to match upstream conditions. By doing this he was able to extend the range over which theory and experiment agreed, but he did not investigate the results in a general context. The distortion of axisymmetric turbulence by a pure straining flow has been studied by Sreenivasan & Narasimha (1978). With $\mathbf{e}$ as unit vector in the preferred direction, the most general form of the spectrum tensor satisfying symmetry and continuity conditions is (Batchelor 1953)

$$\Phi_{ij}(\mathbf{m}) = I_{ij} B_1(m, \mathbf{m}.\mathbf{e}) + H_{ij} B_2(m, \mathbf{m}.\mathbf{e}), \tag{2.4}$$

where

$$I_{ij} = \delta_{ij} - \frac{m_i m_j}{m^2}, \tag{2.5}$$

$$H_{ij} = e_i e_j + \frac{(m_k e_k)^2}{m^2}\delta_{ij} - \frac{m_k e_k (e_i m_j + e_j m_i)}{m^2}. \tag{2.6}$$

For isotropic turbulence B_2 is zero and B_1 is a function of m only, related to the usual energy spectrum function by

$$B_1(m) = \frac{E(m)}{4\pi m^2}. \tag{2.7}$$

There is a wide variety of ways of prescribing the functions B_1 and B_2, which have been enumerated by Herring (1974). The simplest assumption will be made here, namely that B_1 and B_2 are functions of m only and independent of $\mathbf{m}.\mathbf{e}$, which corresponds to Sreenivasan's ansatz I. The physical meaning of this assumption will be discussed shortly. The detailed forms of B_1 and B_2 are not important to the results presented here since the velocity moments depend only on

$$\tilde{B}_n = \frac{8\pi}{3u_0^2}\int_0^\infty m^2 B_n(m)\, dm \quad (n = 1, 2). \tag{2.8}$$

Taking the x_1 axis as the preferred direction, the velocity moments of the undistorted axisymmetric turbulence are

$$\overline{u_1^2} = u_0^2(\tilde{B}_1 + \tilde{B}_2), \tag{2.9}$$

$$\overline{u_2^2} = \overline{u_3^2} = u_0^2(\tilde{B}_1 + \tfrac{1}{2}\tilde{B}_2), \tag{2.10}$$

with zero shear stresses. This may be rewritten by defining the initial value of $\overline{u_1^2}$ as the reference u_0^2, and the ratio of initial moments

$$S = \frac{\overline{u_1^2}}{\overline{u_2^2}} = \frac{\overline{u_1^2}}{\overline{u_3^2}}, \tag{2.11}$$

so that

$$\tilde{B}_1 = \frac{2}{S} - 1, \quad \tilde{B}_2 = 2\left(1 - \frac{1}{S}\right). \tag{2.12), (2.13}$$

This form of axisymmetric turbulence is the simplest form allowing an increase in the $\overline{u_1^2}$ component over $\overline{u_2^2}$ and $\overline{u_3^2}$ while producing minimum change in the spatial structure, since continuity conditions do not allow a simple rescaling of the spectral components. Briefly, it may be shown that introducing $B_2(m)$ raises the mean-square vorticity in the x_2 and x_3 directions over that in the x_1 direction. The return flow for u_2 motions is primarily then in the x_1 direction rather than being equally in the x_1 and x_3 directions, as may be seen by considering the integral lengthscales. Similarly u_3 return flow is primarily in the x_1 direction. This is consistent with an increase in eddy motions in planes with normals perpendicular to the axis of symmetry. If the integral lengthscales $L_{ij}^{(n)}$ are defined by

$$\overline{u_i u_j}\, L_{ij}^{(n)} = \int_0^\infty \overline{u_i(\mathbf{x}, t)\, u_j(\mathbf{x}+r\mathbf{j}^{(n)}, t)}\, dr, \tag{2.14}$$

where $\mathbf{j}^{(n)}$ is a unit vector of the x_n co-ordinate axis, then for this axisymmetric spectrum the integral scales $L_{11}^{(1)}$, $L_{22}^{(2)}$, $L_{33}^{(3)}$ are still equal as for isotropic turbulence, and $L_{11}^{(2)}$, $L_{11}^{(3)}$ are still $\frac{1}{2}L_{11}^{(1)}$. For the special case of $B_2(m)$ proportional to $B_1(m)$, the scales $L_{22}^{(1)}$ and $L_{22}^{(3)}$ though are

$$L_{22}^{(1)} = L_{11}^{(1)}(1-\tfrac{1}{2}S), \tag{2.15}$$

$$L_{22}^{(3)} = \tfrac{1}{2}L_{11}^{(1)}\, S, \tag{2.16}$$

indicating a relative increase in eddying motions and return flow in the Ox_1x_2 plane. Similar results apply for the scales $L_{33}^{(n)}$. It is unrealistic to consider arbitrary values of S. The spectrum tensor given by (2.4) must correspond to a non-negative Hermitian form for it to be realiable and this restricts B_1 to being a non-negative function. So necessarily S has an upper value of 2. Results will be given for S in the range $1 \leqslant S \leqslant 2$.

2.1. *Velocity moments*

Values for the velocity moments $\overline{u_i u_j}$ have been computed from (2.2) using the above spectrum for initially axisymmetric turbulence. The results are given in figures 1 and 2 as functions of the distortion strain α and for various values of initial anisotropy S. The velocity moments have been scaled by the corresponding value of $\overline{q^2}(\alpha)$ so as to eliminate the arbitrary energy level and to show the relative contribution of each of the three velocity components to the total turbulent kinetic energy. The results show that shear distortion leads to a distribution of energy with $\overline{u_1^2} > \overline{u_2^2} > \overline{u_3^2}$ essentially. The effect of axial symmetry in the initial conditions is to reduce the relative strength of the shear stress ratio $-\overline{u_1u_3}/\overline{q^2}$ and to raise the relative level of $\overline{u_2^2}$ at the expense of $\overline{u_1^2}$. The fraction of energy in the u_3 component is also reduced somewhat by the effect of initial conditions. Figure 3 shows further ratios of the Reynolds shear stress. In particular the initial anisotropy S reduces the maximum value of $\tau/\rho u_1' u_3'$ from 0·74 for initially isotropic turbulence to 0·42 for $S = 2$, again reflecting the lower level of Reynolds shear stress for this form of initially axisymmetric turbulence.

For small strains the velocity moments, without scaling, are

$$\overline{u_1^2} = \overline{u_1^2}(\alpha=0)\left[1+\tfrac{1}{35}\alpha^2\left(\frac{19}{S}-9\right)+O(\alpha^4)\right], \tag{2.17}$$

$$\overline{u_2^2} = \overline{u_1^2}(\alpha=0)\left[\frac{1}{S}+\tfrac{1}{35}\alpha^2\left(\frac{4}{S}-4\right)+O(\alpha^4)\right], \tag{2.18}$$

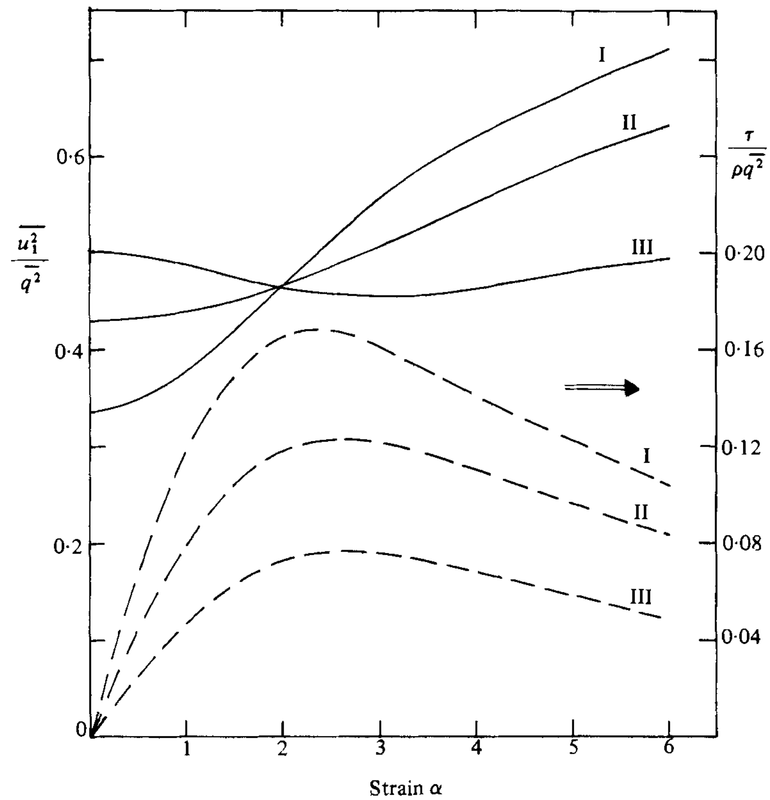

FIGURE 1. Stress ratios calculated from rapid-distortion theory for initially axisymmetric turbulence: ——, $\overline{u_1^2}/\overline{q^2}$; – – – –, $-\overline{u_1 u_3}/\overline{q^2}$. Initial turbulence: I, $S = 1$; II, 1·5; III, 2·0, where $S = \overline{u_1^2}/\overline{u_2^2}$ at zero strain. $\overline{q^2} = \overline{u_k u_k}$.

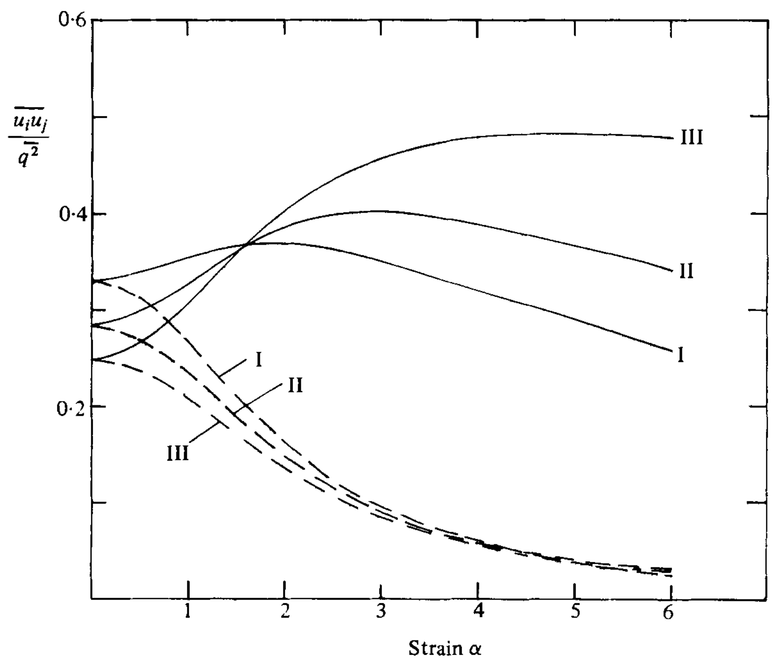

FIGURE 2. Stress ratios calculated from rapid-distortion theory: ——, $\overline{u_2^2}/\overline{q^2}$; – – – –, $\overline{u_3^2}/\overline{q^2}$.

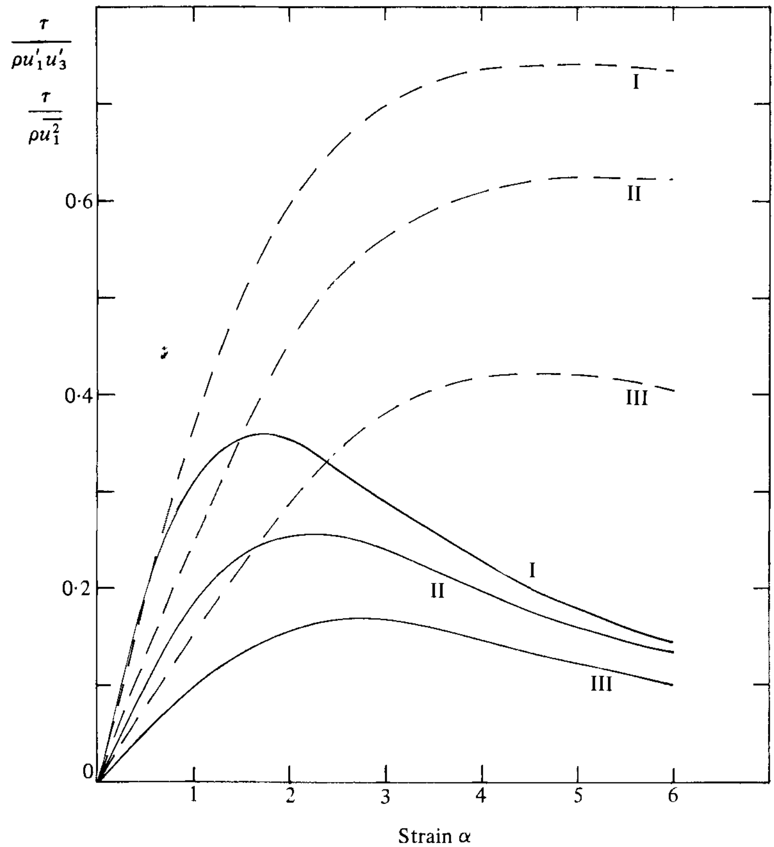

FIGURE 3. Reynolds-stress ratios calculated from rapid-distortion theory: ——, $\tau/\rho\overline{q^2}$; ----, $\tau/\rho u_1' u_3'$. Initial turbulence: I, $S = 1$; II, 1·5; III, 2.

$$\overline{u_3^2} = \overline{u_1^2}(\alpha = 0)\left[\frac{1}{S} - \tfrac{1}{35}\alpha^2\left(\frac{2}{S}+2\right) + O(\alpha^4)\right], \tag{2.19}$$

$$\overline{-u_1 u_3} = \overline{u_1^2}(\alpha = 0)\left[\tfrac{1}{5}\alpha\left(\frac{3}{S}-1\right) + O(\alpha^5)\right]. \tag{2.20}$$

Further insight into the production and distribution of energy may be obtained by examining the pressure–strain-rate correlations.

2.2. *Pressure–strain-rate correlations*

The rapid-distortion problem may be written in terms of a set of moment transport equations with nonlinear effects, viscosity, and energy decay neglected:

$$\frac{\partial \overline{u_1^2}}{\partial t} = -2\overline{u_1 u_3}\beta(t) + \sigma_{11}, \tag{2.21}$$

$$\frac{\partial \overline{u_2^2}}{\partial t} = \sigma_{22}, \tag{2.22}$$

$$\frac{\partial \overline{u_3^2}}{\partial t} = \sigma_{33}, \tag{2.23}$$

$$\frac{\partial \overline{u_1 u_3}}{\partial t} = -\overline{u_3^2}\beta(t) + \sigma_{13}. \tag{2.24}$$

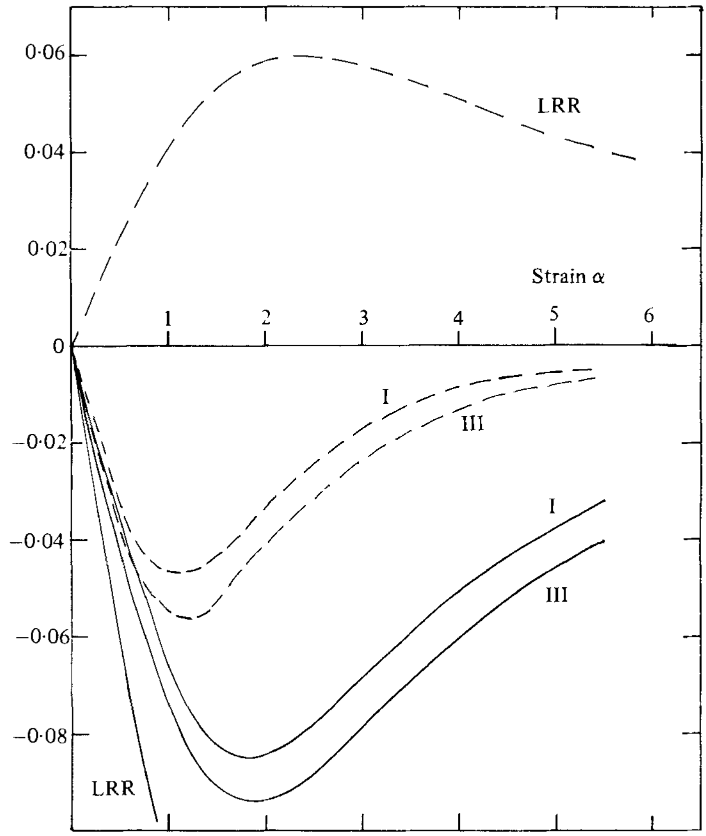

FIGURE 4. Pressure-strain rate correlations calculated from rapid distortion theory: ——, $\sigma_{11}/\beta\overline{q^2}$; ----, $\sigma_{33}/\beta\overline{q^2}$; and comparison with corresponding estimates from LRR model for initially isotropic turbulence. Initial turbulence: I, $S = 1$; III, 2. Values for $S = 1{\cdot}5$ were intermediate between curves I and III.

The pressure–strain-rate tensor σ_{ij} is defined to be

$$\sigma_{ij} = \overline{\frac{p}{\rho}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\}}, \tag{2.25}$$

and may be calculated for the rapid-distortion problem from the results given in the appendix and from the chosen form of the initial spectrum tensor. The fluctuating pressure and hence the pressure–strain-rate correlations are both directly proportional to the mean shear $\beta(t)$. The correlations σ_{ij} have been scaled by $\beta(t)\overline{q^2}$ and are shown in figures 4 and 5. As for the velocity moments, the details of B_1 and B_2 are not important. The integral (A 16) for σ_{ij} in the appendix depends on m only through the initial spectrum tensor Φ_{ij} and for the particular choice of initial axisymmetric turbulence made here this reduces to a dependence only on $\tilde{B}_1$ and $\tilde{B}_2$.

Without the effect of pressure–strain-rate terms, (2.21)–(2.23) show that turbulent-energy production would be solely concentrated in the $\overline{u_1^2}$ component. The pressure–strain-rate interactions immediately on application of the strain give a net transfer of energy between the velocity components. Their net effect is zero, i.e. the sum of σ_{11}, σ_{22} and σ_{33} is zero for incompressible flows. The values plotted in figures 4 and 5 show that σ_{11} and σ_{33} are negative while σ_{22} is positive, indicating that energy is

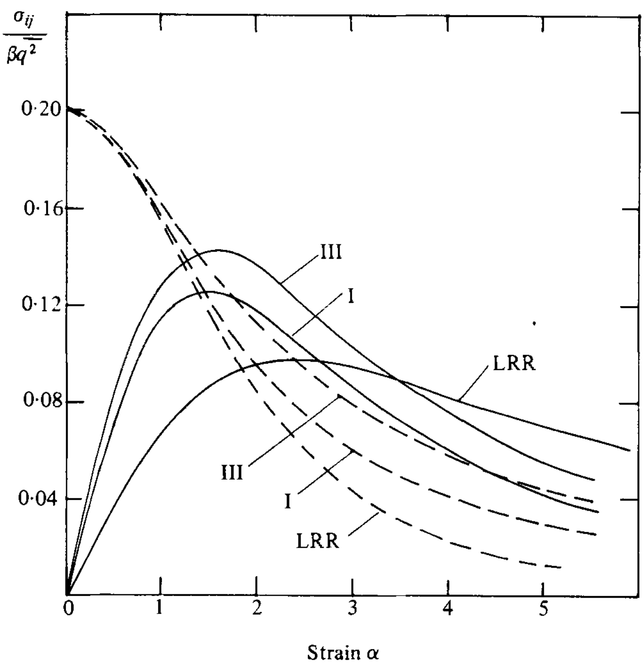

FIGURE 5. Pressure–strain-rate conditions calculated from rapid-distortion theory: ——, $\sigma_{22}/\beta\overline{q^2}$; - - - - -, $\sigma_{13}/\beta\overline{q^2}$. See caption to figure 4.

transferred from both u_1 and u_3 motions to the transverse direction u_2. This accounts for the large reduction in $\overline{u_3^2}/\overline{q^2}$ by the shear distortion. A positive shear stress is generated by the mean shear, but its magnitude is limited by the effects of σ_{13}. Further, from (2.24) the value of the shear stress is determined by the level of $\overline{u_3^2}$ rather than $\overline{u_1^2}$. For initially axisymmetric turbulence $\overline{u_3^2}$ is lower and σ_{13} larger, both effects tending to reduce the level of the Reynolds shear stress. The overall effect of initial axial symmetry is to raise the relative magnitude of all the components of the pressure–strain-rate correlation and so accentuate their influence. For small values of the distortion strain $\alpha(t)$ the unscaled non-zero components of σ_{ij} are

$$\sigma_{11} = \beta(t)\,\overline{u_1^2}(\alpha = 0)\left[-\tfrac{2}{35}\alpha\left(1+\frac{1}{S}\right)+\tfrac{1}{3465}\alpha^3\left(64-\frac{78}{S}\right)+O(\alpha^5)\right], \tag{2.26}$$

$$\sigma_{22} = \beta(t)\,\overline{u_1^2}(\alpha = 0)\left[\tfrac{4}{35}\alpha\left(1+\frac{1}{S}\right)-\tfrac{4}{3465}\alpha^3\left(43-\frac{6}{S}\right)+O(\alpha^5)\right], \tag{2.27}$$

$$\sigma_{33} = \beta(t)\,\overline{u_1^2}(\alpha = 0)\left[-\tfrac{2}{35}\alpha\left(1+\frac{1}{S}\right)+\tfrac{54}{3465}\alpha^3\left(2+\frac{1}{S}\right)+O(\alpha^5)\right], \tag{2.28}$$

$$\sigma_{13} = \beta(t)\,\overline{u_1^2}(\alpha = 0)\left[\frac{1}{10}\left(1+\frac{2}{S}\right)-\tfrac{1}{35}\alpha^2\left(1+\frac{1}{S}\right)+O(\alpha^4)\right]. \tag{2.29}$$

The pressure–strain-rate correlations play an important role in moment-transport-equation models of turbulence, and as mentioned earlier much effort has gone into approximating them in terms of instantaneous values of the velocity moments and turbulence dissipation rate. When nonlinear processes are included and the full equations of motion considered, the pressure has two components: one dependent on the fluctuating values of $u_i u_j$ and the other linear in the mean shear. The pressure–

strain-rate tensor then has two corresponding parts: $\sigma_{ij}^{(N)}$ representing nonlinear effects, and $\sigma_{ij}^{(L)}$ containing the pressure component proportional to the mean shear. The nonlinear part is identified with a tendency for the turbulence to return to isotropy, and is usually approximated by Rotta's (1951) hypothesis:

$$\sigma_{ij}^{(N)} = -2c_1\epsilon\left(\frac{\overline{u_i u_j}}{\overline{q^2}}-\tfrac{1}{3}\delta_{ij}\right), \tag{2.30}$$

in the notation of Launder *et al.* (1975). The linear part referred to as the rapid term by Lumley (1978) is approximated by a tensor equation linear in the mean shear and velocity moments. The usual assumption is that, at least for weakly distorted turbulence, the tensor equation is isotropic. For uniform shear flow this leads to the approximations

$$\sigma_{11}^{(L)} = -\tfrac{1}{11}\overline{u_1u_3}\,\beta(2c_2-6), \tag{2.31}$$

$$\sigma_{22}^{(L)} = -\tfrac{1}{11}\overline{u_1u_3}\,\beta(3c_2+2), \tag{2.32}$$

$$\sigma_{33}^{(L)} = -\tfrac{1}{11}\overline{u_1u_3}\,\beta(4-5c_2), \tag{2.33}$$

$$\sigma_{13}^{(L)} = \tfrac{1}{110}\beta[\overline{u_1^2}(25c_2-9)+\overline{u_2^2}(1-15c_2)+\overline{u_3^2}(41-10c_2)], \tag{2.34}$$

again in the notation of Launder *et al.* (1975).

In the limit of rapid distortion these approximate estimates of $\sigma_{ij}^{(L)}$ can be compared with the results calculated directly from rapid-distortion theory. The values of $\overline{u_iu_j}$ in (2.31)–(2.34) are taken from rapid-distortion theory on the basis of initially isotropic turbulence ($S = 1$) and the value of c_2 is taken to be 0·4. The results are included in figures 4 and 5. For zero strain when the turbulence is isotropic the estimates of $\sigma_{ij}^{(L)}$ do agree with the rapid-distortion values of σ_{ij}, independently of the choice of c_2. However, once the turbulence is distorted by any amount, $\sigma_{ij}^{(L)}$ does not provide a good estimate of σ_{ij}. The component $\sigma_{33}^{(L)}$ is of the opposite sign, corresponding to a transfer of energy to u_3, and the other components differ noticeably. A value of c_2 in excess of 0·8 would be required to make $\sigma_{33}^{(L)}$ negative and of the same sign as σ_{33}.

Gence, Angel & Mathieu (1978) compared rapid-distortion estimates for initially isotropic turbulence of pressure–strain-rate correlations with the models used in moment-transport equations. They found that better agreement between $\sigma_{ij}^{(L)}$ and σ_{ij} could be obtained by choosing $c_2 = 1{\cdot}53$ for weak distortions or $c_2 = 1{\cdot}13$ as a general compromise value.† They based their results on both shear distortion and distortion in pure straining flows. They also proposed a modified formula for $\sigma_{ij}^{(L)}$ to remove the discrepancies.

For the more general case of initially axisymmetric turbulence there is really no suitable value of c_2 that will match the estimates of $\sigma_{ij}^{(L)}$ to the results of rapid distortion. This may be seen in particular by comparing these estimates (2.31)–(2.34) with the expansions of σ_{ij} for small distortion strains given by (2.26)–(2.29) and the velocity-moment expansions (2.17)–(2.20). At zero distortion strain even $\sigma_{13}^{(L)}$ as estimated by (2.34) fails to match the correct value in general unless c_2 is set to equal 0·8. If this value of c_2 is adopted then $\sigma_{33}^{(L)}$ is identically zero.

While it is reasonable to suppose that the moments $\overline{u_iu_j}$ and the turbulent-energy dissipation ϵ characterize the turbulence and in particular the pressure–strain-rate correlations, there is no *a priori* reason for this in principle, and certainly no special reason to expect a linear isotropic tensor equation. For initially isotropic turbulence

† In Gence's notation $D = -\frac{1}{11}(2+3c_2)$, and values of $D = -0{\cdot}6$ or $-0{\cdot}49$ were suggested.

at zero strain the approximations for $\sigma_{ij}^{(L)}$ do match σ_{ij} in the rapid distortion limit, but the tensor equation does not match rapid distortion in general, even for initially isotropic turbulence once the strains are $O(1)$. The integral expressions (A 14) and (A 16) in the appendix for $\overline{u_i u_j}$ and σ_{ij} show their distinct character and separate dependence on the initial spectrum tensor. Also the results depend on strain α, i.e. the history of the mean shear, and not just instantaneous values.

For equilibrium shear flows the two parts $\sigma_{ij}^{(L)}$ and $\sigma_{ij}^{(N)}$ are of similar magnitude, so that discrepancies in estimating one may be compensated by the other, with little overall difference. In an experiment it will be very difficult to determine the two parts separately. For a rapidly changing flow the differences in estimating the rapid part of the pressure–strain-rate correlation will be significant.

2.3. *Integral lengthscales*

The integral lengthscales defined by (2.14) have been evaluated for the distorted turbulence

$$\overline{u_1^2}\,L_{11}^{(1)} = [\overline{u_1^2}\,L_{11}^{(1)}]\,(\alpha = 0)\,(1+\tfrac{1}{2}B_1^+\alpha^2), \tag{2.35}$$

$$\overline{u_2^2}\,L_{22}^{(1)} = [\overline{u_1^2}\,L_{11}^{(1)}]\,(\alpha = 0)\,(\tfrac{1}{2}B_1^+), \tag{2.36}$$

$$\overline{u_2^2}\,L_{22}^{(2)} = [\overline{u_1^2}\,L_{11}^{(1)}]\,(\alpha = 0)\,(1-\tfrac{1}{2}B_1^+), \tag{2.37}$$

$$\overline{u_3^2}\,L_{33}^{(1)} = [\overline{u_1^2}\,L_{11}^{(1)}]\,(\alpha = 0)\,(\tfrac{1}{2}B_1^+), \tag{2.38}$$

$$\overline{u_3^2}\,L_{33}^{(3)} = [\overline{u_1^2}\,L_{11}^{(1)}]\,(\alpha = 0)\,(1-\tfrac{1}{2}B_2^+)\,(1+\alpha^2)^{\frac{1}{2}}. \tag{2.39}$$

The coefficients B_1^+, B_2^+ are moments of the spectral functions $B_1(m)$ and $B_2(m)$. If B_1 and B_2 are assumed to be proportional then B_i^+ is to equal $\tilde{B}_i$, as in the velocity moments, otherwise

$$B_i^+[\overline{u_1^2}\,L_{11}^{(1)}]\,(\alpha = 0) = 2\pi^2\int_0^\infty mB_i(m)\,dm. \tag{2.40}$$

The results show that $L_{11}^{(1)}$ and $L_{33}^{(3)}$ increase while $L_{22}^{(2)}$ decreases because of the formation of a negative loop in the u_2 space correlation. Similarly $L_{33}^{(1)}$ rises and $L_{22}^{(1)}$ decreases. These changes in lengthscales may be understood more easily by considering the changes in vorticity components and the typical pattern of eddy motions after distortion.

The equation for vorticity fluctuations corresponding to the rapid-distortion approximation made here is

$$\frac{\partial\omega_i}{\partial t}+\beta x_3\frac{\partial\omega_i}{\partial x_1} = \beta\frac{\partial u_i}{\partial x_2}+\beta\omega_3\,\delta_{i1}. \tag{2.41}$$

The terms on the right-hand side represent respectively the effects of mean vorticity stretching and the rotation of fluctuating vorticity by the mean shear. As described by Hunt (1978), an ω_3 component of vorticity generated by mean vorticity stretching will be rotated by the mean flow, giving a ω_1 component and a positive Reynolds shear stress. The hairpin vortex is a typical resulting eddy structure, similar to that proposed by Theodorsen (1952). The decreases in $L_{22}^{(2)}$ and $L_{22}^{(1)}$ are consistent with an increase in ω_1 and ω_3 vorticity levels and with greater return flow for u_2 motions in the x_1 direction. The rises in $L_{11}^{(1)}$, $L_{33}^{(1)}$, and $L_{33}^{(3)}$ are consistent with a relative suppression of ω_2 vorticity and reduced return flow for u_3 in the x_1 direction. The effect of initial axial symmetry is to reduce $L_{22}^{(1)}$ and $L_{22}^{(2)}$ further and to limit the increase of $L_{33}^{(1)}$ and $L_{33}^{(3)}$. The initially higher levels of ω_2, ω_3 vorticity for axisymmetric turbulence tend to raise the importance of ω_3 vorticity and counter the effects of distortion on the two other vorticity components.

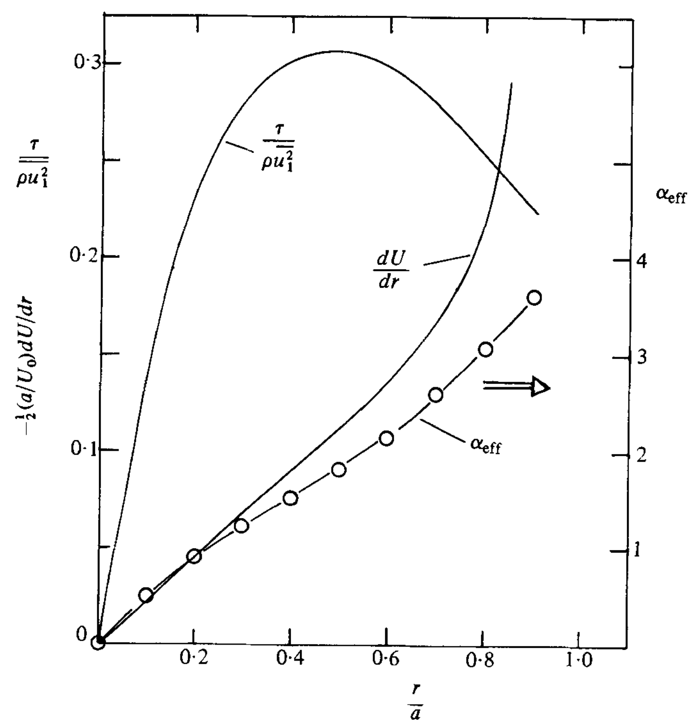

FIGURE 6. Data from Laufer (1954) for flow in a pipe: observed mean shear and stress ratio. Effective strain α derived by comparison of stress ratio with rapid distortion for $S = 1 \cdot 10$. $Re = 500000$.

3. Comparisons with some simple shear flows

The results of §2 were derived on the assumption of a rapid distortion of homogeneous turbulence. For steady equilibrium shear flows in pipes and channels this assumption is clearly not valid, and the ability of rapid-distortion theory to describe the structure of simple shear flows, as demonstrated by Townsend (1970), is to some extent surprising. However, rapid distortion will describe the initial development or establishment of turbulence structure in a shear flow, and nonlinear processes seem to tend to limit this development rather than radically alter the structure. In a pipe or channel flow the mean shear will vanish at the centreline and increase towards the walls. If the turbulence is regarded as being approximately homogenous locally then the distortion strain at the centre should be zero, for a given time interval, and increase towards the wall. The stress ratio $\tau/\rho\overline{u_1^2}$ calculated from rapid-distortion theory (figure 3, $S = 1$) as a function of increasing strain shows a close similarity with the same ratio as a function of distance from the centreline derived from the experimental results of Laufer (1954) for flow in a pipe (figure 6), and, further, both curves have approximately the same maximum values. This suggests the possibility of ascribing a distribution of effective distortion strain as a function of distance from the centreline based on the values of this stress ratio. This is the basis of the approach taken here. The developed turbulence structure is supposed to be similar to that locally of a truncated rapid distortion, stopped when the distortion strain has settled to a local equilibrium value that will be referred to as the equilibrium value of the effective distortion strain. Velocity moments will be compared between rapid distortion and observational data to determine effective-distortion-strain values from $\tau/\rho\overline{u_1^2}$ values and then to test the overall usefulness of the hypothesis.

Before comparing the results of rapid distortion with pipe or channel flows an assumption of local homogeneity must be made so that the mean shear is assumed to be locally uniform on the scale of the most energetic turbulent eddies, an assumption also invoked by moment-transport-equation models. The validity of this type of assumption has been questioned though by Corrsin (1957). Further, for equilibrium shear flows the non-linear processes previously neglected should be considered for their possible effects. Briefly some of these effects will include self-advection of the turbulence, random distortion of turbulent eddies leading to energy transfer from large scales to small scales and breakdown of individual eddies, and the additional contribution of random pressure fluctuations. Turbulent self-advection is the essential feature of turbulent diffusion, and in inhomogeneous flows will lead to diffusion of turbulent kinetic energy. It will also lead to the migration of turbulent eddies from one part of the flow to another so that eddies experience differing mean shears during their lifetime. At a fixed point in the flow eddies with different strain histories would be observed as they migrate from neighbouring regions. The net effect may be regarded as producing a transport of effective strain.

The transfer of energy from large scales to smaller scales is the principal mechanism for energy loss since the direct effect of viscosity on the larger scales will in general be small. The experiments of Comte-Bellot & Corrsin (1966) on the decay of homogeneous isotropic turbulence, for example, show that the energy decays on a timescale comparable to the integral timescale

$$\frac{\partial\overline{q^2}}{\partial t} = \frac{-\overline{q^2}}{T_E}, \tag{3.1}$$

$$T_E = \frac{L}{\eta(\overline{q^2})^{\frac{1}{2}}}, \tag{3.2}$$

where L is an integral lengthscale and η a constant. If the timescale T_E is also taken to be a typical time before an eddy feature is broken down into smaller-scale motions, this will set a time limit for the interval for which the eddy will be distorted by the effect of mean shear. For a uniform shear an equation analogous to (3.1) will then apply to the effective strain, modifying the original form of (2.3),

$$\frac{\partial\alpha_{\mathrm{eff}}}{\partial t} = \frac{\partial U_1}{\partial x_3} - \frac{\alpha_{\mathrm{eff}}}{T_D}. \tag{3.3}$$

The distortion timescale

$$T_D = \frac{L}{\delta(\overline{q^2})^{\frac{1}{2}}}, \tag{3.4}$$

where δ is another constant. In the limit of a rapid distortion the above estimate for effective strain will revert to the usual definition of distortion strain (2.3), while in steady equilibrium shearing the effective strain will have a limit of $T_D\,\partial U_1/\partial x_3$. Finally there is the effect of the nonlinear pressure fluctuations that contribute to the pressure–strain-rate correlations $\sigma_{ij}^{(N)}$ discussed earlier (equation (2.30)). These will tend to reduce the anisotropy of the turbulence over and above the effect of a limited straining time.

Some of the nonlinear effects may then be accounted for by specifying diffusion terms and decay timescales. No specific correction for pressure effects has been

Experiment	Ratio S	max $(\tau/\rho\overline{u_1^2})$	max $(\tau/\rho\overline{q^2})$	T_D
(A) Laufer (1954), pipe,				
($A1$) $Re = 50000$	1·10	0·33	0·16	—
($A2$) $Re = 500000$	1·10	0·31	0·145	$0{\cdot}28\frac{a}{u_*}$, $7{\cdot}9\frac{a}{U_0}$
(B) Sabot & Comte-Bellot (1976), pipe				
$Re = 135000$	1·24	0·30	—	—
(C) Lawn (1971), pipe				
$Re = 90000$	1·55	0·25	0·13	$0{\cdot}23\frac{a}{u_*}$
(D) Laufer (1951), channel				
$Re = 61600$	1·6	0·25	—	—
(E) Comte-Bellot (1963), channel				
($E1$) $Re = 57000$	1·66	0·19	0·11	—
($E2$) $Re = 120000$	1·8	0·20	0·12	$0{\cdot}3\frac{D}{u_*}$, $8{\cdot}0\frac{D}{U_0}$
($E3$) $Re = 230000$	2·2	0·165	0·105	—

TABLE 1. Centreline values of S and maximum observed stress ratios, with estimates of distortion timescale T_D. Maximum mean velocity U_0; friction velocity u_*; pipe radius a; channel depth, centreline to wall, D

included though. Conventionally, in moment-transport equations for very large Reynolds numbers, the decay of energy is specified as being equally divided among the three components; so for uniform shearing

$$\frac{\partial \overline{u_i u_j}}{\partial t} = -\left(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right) + \sigma_{ij}^{(L)} + \sigma_{ij}^{(N)} - \tfrac{2}{3}\epsilon\delta_{ij}, \tag{3.5}$$

$$\epsilon = \frac{\overline{q^2}}{2T_E}. \tag{3.6}$$

The use of rapid distortion to predict stress ratios is consistent with postulating a Rayleigh damping term $-\overline{u_i u_j}/T_E$ in the velocity-moment equations to replace the nonlinear pressure–strain-rate correlation and decay term,

$$\frac{\partial \overline{u_i u_j}}{\partial t} = -\left(\overline{u_i u_k}\frac{\partial u_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial u_i}{\partial x_k}\right) + \sigma_{ij} - \frac{\overline{u_i u_j}}{T_E}. \tag{3.7}$$

The stress ratios are then unaltered by the decay process. Equation (3.7) would also result from specifying $\sigma_{ij}^{(N)}$ by (2.30) and setting $c_1 = 1{\cdot}0$. The value of c_1 usually recommended is 1·8. So any tendency for equipartion of energy will be underestimated by rapid-distortion theory. The preceding discussion illustrates how nonlinear inertial processes may be expected to limit the distortion effects of a mean shear. The distortion reaches an equilibrium with the structure characterized by a finite asymptotic value of effective distortion strain α_{eff}, based on comparison with the results of rapid-distortion theory, rather than an indefinitely increasing strain α, as given by the usual definition (2.3).

Data from several channel- and pipe-flow experiments have been analysed to evaluate the stress ratios $\tau/\rho\overline{u_1^2}$ and $\tau/\rho\overline{q^2}$ as functions of position. The values of $\overline{u_1^2}$ and $\overline{q^2}$ are taken from the measurements, while the shear stress is based on a linear profile appropriate to the core region of fully developed channel or pipe flow. The distribution of the ratio $\tau/\rho\overline{u_1^2}$ was used to specify a profile of effective strain by

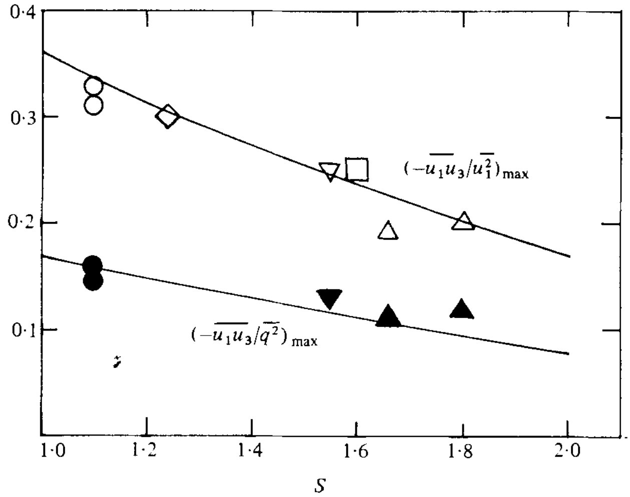

FIGURE 7. Calculated values of maxima of stress ratios $-\overline{u_1u_3}/\overline{u_1^2}$ and $-\overline{u_1u_3}/\overline{q^2}$ as functions of strain, for fixed values of initial anisotropy. Symbols denote observed values of these maxima as listed in table 1.

comparison with the predicted results of rapid-distortion theory. This ratio was chosen as it showed greater variations than $\tau/\rho\overline{q^2}$, and because it had a definite maximum value, thus providing a reference point. Table 1 lists the experiments considered. At the centrelines of these flows the effective strain is zero by symmetry, eddies passing through the centreline experiencing on average equally positive and negative strains, and so the structure at the centreline should characterize the undistorted state of turbulence in the core region. In practice the turbulence at the centreline of two-dimensional channel and pipe flows is not isotropic but more nearly axisymmetric, with $\overline{u_1^2}$ being largest and the moments $\overline{u_2^2}$, $\overline{u_3^2}$ nearly equal. The observations have been compared with results of §2, taking into account this axisymmetric structure for the undistorted turbulence. The ratio S is defined now as the ratio of moments at the centreline:

$$S = [2\overline{u_1^2}/(\overline{u_2^2}+\overline{u_3^2})]\,(x_3 = 0). \tag{3.8}$$

Examination of the experimental data shows that the maximum values of the profiles for the stress ratios $\tau/\rho\overline{u_1^2}$ and $\tau/\rho\overline{q^2}$ differ appreciably between experiments. The maxima of these ratios as functions of distance from the centreline are listed in table 1 and plotted in figure 7 against the corresponding value of S for each experiment. There is a clear trend for the stress ratio values to decrease with increasing S. In the same figure this trend is compared with the variations in maximum values with S predicted by the rapid-distortion results of §2. There is a close similarity, within experimental limits, of the observed trend and the theoretical curves, showing the ability of rapid distortion to give the correct values and indicating the importance of the parameter S. There is no clear-cut explanation for the observed variations in S. Examination of the experiments listed here and other experiments suggest though that S has a value of about 1·2 for pipe flows and about 1·8 for channel flows.

In figures 6, 8 and 9 three different flows have been analysed for profiles of the stress ratio $\tau/\rho\overline{u_1^2}$ and for profiles of the mean shear. Values of effective strain α_{eff} have then

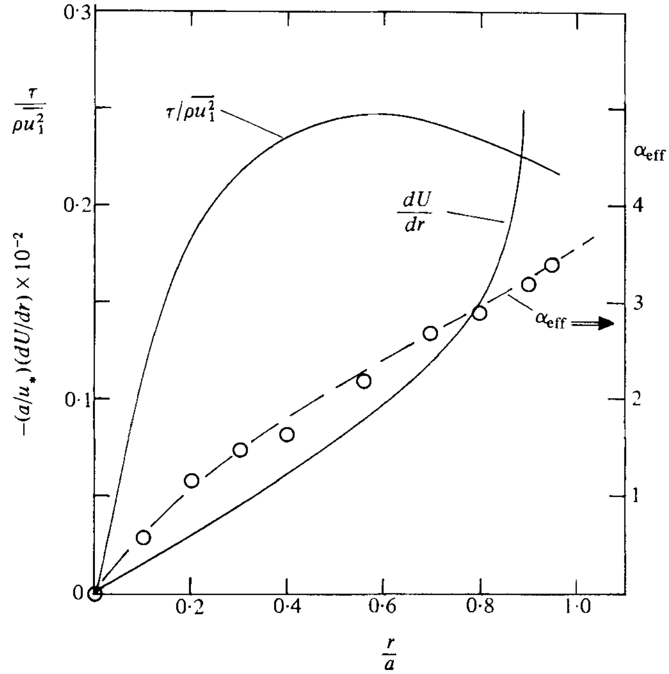

FIGURE 8. Data from Lawn (1971) for flow in a pipe: observed mean shear and stress ratio. Effective strain α derived by comparison of stress ratio with rapid distortion for $S = 1{\cdot}55$. $Re = 90000$.

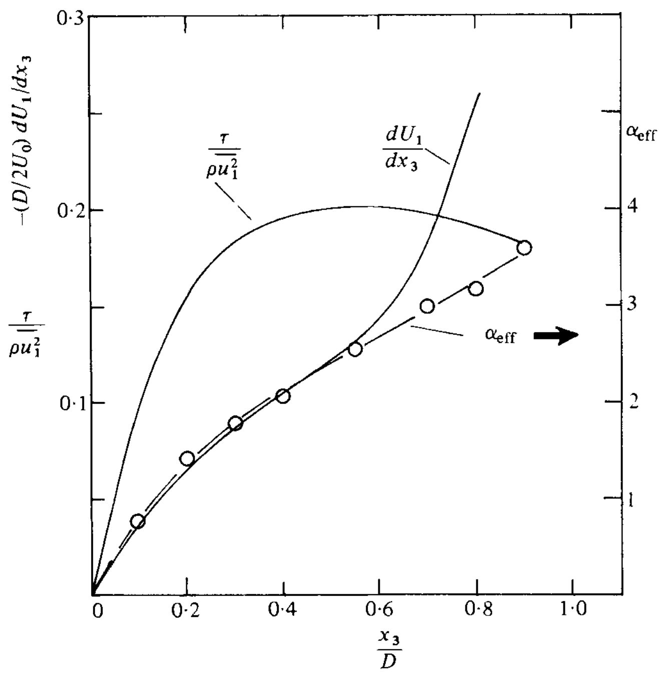

FIGURE 9. Data from Comte-Bellot (1963) for flow in a channel: observed mean shear and stress ratio. Effective strain α derived by comparison of stress ratio with rapid distortion for $S = 1{\cdot}8$. $Re = 120000$.

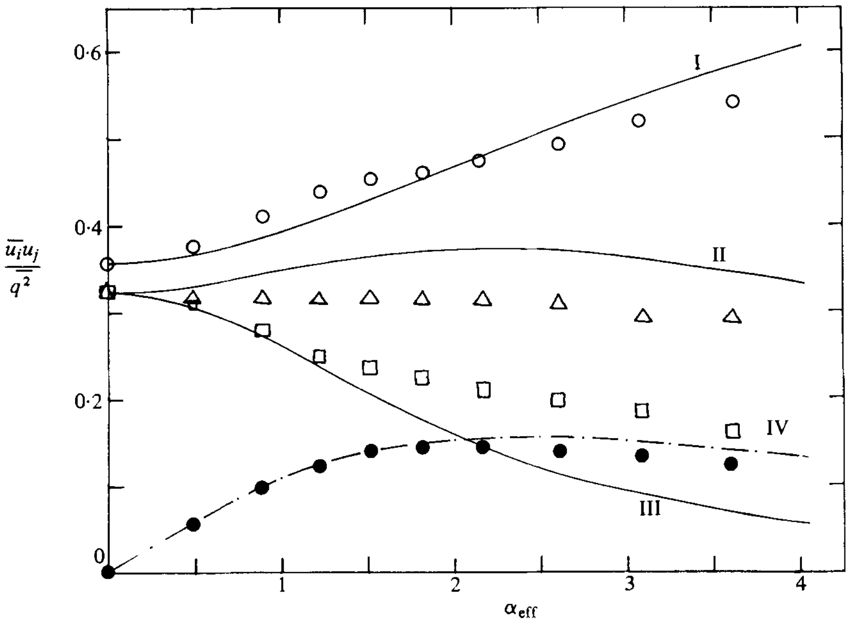

FIGURE 10. Comparison of the observed stress ratios $\overline{u_iu_j}/\overline{q^2}$ from Laufer (1954) with rapid distortion ($S = 1{\cdot}1$) based on values of effective strain. Ratio $\overline{u_1^2}/\overline{q^2}$: curve I from theory; ○, observed. Ratio $\overline{u_2^2}/\overline{q^2}$: curve II from theory; △, observed. Ratio $\overline{u_3^2}/\overline{q^2}$: curve III from theory; □, observed. Ratio $-\overline{u_1u_3}/\overline{q^2}$: —·—·—, from theory; ●, observed.

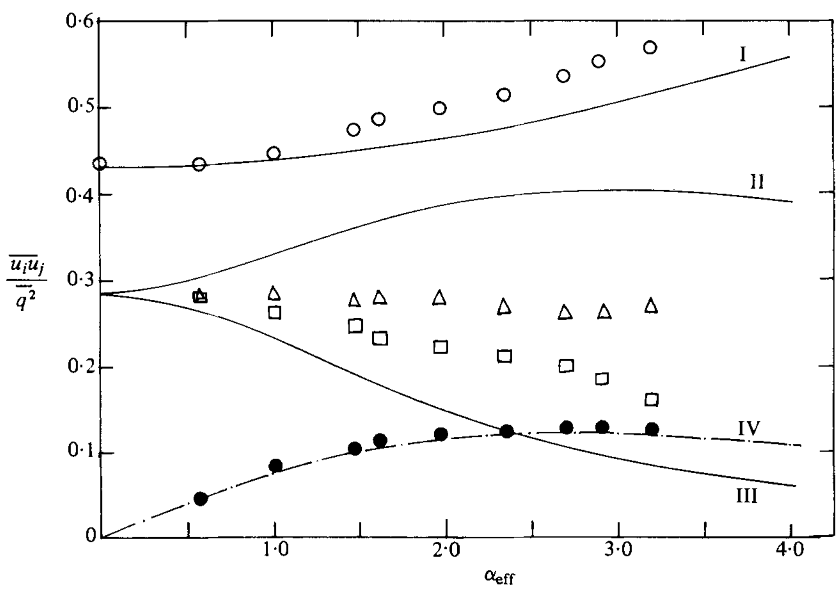

FIGURE 11. Comparison of the observed stress ratios $\overline{u_iu_j}/\overline{q^2}$ from Lawn (1971) with rapid distortion ($S = 1{\cdot}55$) based on values of effective strain. Symbols as in figure 10.

been derived by comparing observations with rapid distortion so as to match the stress ratio. The profiles of effective strain in all three cases are similar, varying roughly linearly over the core region from zero to 3·5 or 4. The mean-shear profile varies linearly over a central region, and the ratio of mean shear to effective strain in this region gives an estimate of the distortion timescale T_{D} when diffusion of strain is neglected. The values of T_{D} listed in table 1 are of similar magnitude, and also comparable to the interval between bursts in pipe flow observed by Sabot & Comte-Bellot (1976), and with the energy decay timescale T_{E} for this region. The profiles of effective strain have then been used to compare the other moment ratios with rapid distortion estimates as given in figures 10 and 11. The ratios are predicted approximately by the theory, but most obviously the difference between $\overline{u_2^2}$ and $\overline{u_3^2}$ is overestimated, indicating the importance of a return-to-isotropy effect.

Previous authors such as Deissler (1975) and Loiseau (1973) have compared rapid-distortion theory with the experiments on homogeneous shear flow by Champagne *et al.* (1970), Harris, Graham & Corrsin (1977) and Mulhearn & Luxton (1975). The experiments were performed to obtain an asymptotic equilibrium structure for the turbulence in mean shear. The previous theoretical comparisons mentioned above assumed, however, that this asymptotic structure would come from taking the limit of large distortion strain as defined by (2.3), rather than the distortion process being limited by a finite distortion timescale giving a finite effective distortion strain. The experimental data of Harris *et al.* (1977, figure 5) show clearly though that $\overline{q^2}/\beta^2 L_1^2$ is tending to an asymptotic limit consistent with the effective strain distortion α_{eff} reaching an upper limit. The most recent set of data of Tavoularis & Corrsin (1981) shows that $\tau/\rho\overline{u_1^2}$ starts with a value of about 0·34 at $x_1/h = 4$ and settles downstream to an asymptotic value of 0·27, while $\tau/\rho\overline{q^2}$ tends to 0·14. The ratios of velocity moments tend to level out at

$$\frac{\overline{u_1^2}}{\overline{q^2}}:\frac{\overline{u_2^2}}{\overline{q^2}}:\frac{\overline{u_3^2}}{\overline{q^2}} = 0{\cdot}53:0{\cdot}28:0{\cdot}19, \tag{3.9}$$

in the present notation. This is consistent with initially isotropic turbulence being distorted to a finite strain of 3·5 or so, for which $\tau/\rho\overline{q^2}$ is 0·14, $\tau/\rho\overline{u_1^2}$ is 0·26 and the ratios of velocity moments are

$$\frac{\overline{u_1^2}}{\overline{q^2}}:\frac{\overline{u_2^2}}{\overline{q^2}}:\frac{\overline{u_3^2}}{\overline{q^2}} = 0{\cdot}59:0{\cdot}34:0{\cdot}07. \tag{3.10}$$

As with the pipe and channel flows, the component $\overline{u_3^2}/\overline{q^2}$ is underestimated and $\overline{u_2^2}/\overline{q^2}$ overestimated by rapid-distortion estimates.

4. Conclusions

The main conclusion is that turbulent shear flows can be reasonably well described by rapid-distortion results in terms of an effective strain, which under normal conditions has an upper value of about 3·5. The two limits of rapidly evolving flows and equilibrium flows are related through an effective-strain equation such as (3.3), of which the important feature is the relaxation timescale T_{D}. The correspondence is by no means exact, but does provide a convenient way of characterizing the turbulence structure, at least to a first approximation, and showing up those features that may be expected as a 'linear' response to the mean shear. It also provides a context for relating the observations of homogeneous shear-flow experiments to the equilibrium structure of turbulent channel or pipe flow. It indicates that even in a fully developed flow there is a full range of turbulence structure going from the unstrained condition to the asymptotic upper value.

For an inhomogeneous flow the effective-strain equation should be modified to allow for the transport of effective strain by the advection of turbulent eddies by the turbulence itself. Such a modification would be to specify a diffusion coefficient E_D:

$$\frac{\partial \alpha_{\text{eff}}}{\partial t} = \frac{\partial u_1}{\partial x_3} + \frac{\partial}{\partial x_3}\left(E_D \frac{\partial \alpha_{\text{eff}}}{\partial x_3}\right) - \frac{\alpha_{\text{eff}}}{T_D}. \tag{4.1}$$

A diffusion-of-strain equation was proposed by Townsend (1970), although without a relaxation term. This omission leads to unrealistically large values of effective strain.

An effective-strain equation also provides the basis for a turbulence model for computing both steady and rapidly varying flows. The effective strain can be used to determine the stress ratio $\tau/\rho\overline{q^2}$, which in turn can be used to estimate turbulence production in an energy equation. The combination of energy equation, strain equation and mean-momentum equation provide a closed model that will reproduce as much detail of the velocity moments as moment-transport equations, yet with fewer equations to solve. The model further does not assume a local time dependence and so is particularly suited to studying oscillating turbulent flows where at higher frequencies the oscillation period is of similar magnitude as the large-eddy timescale. The model has been applied to this problem by Hunt & Maxey (1980) and a subsequent paper to the present one will give a more complete study.

The idea of a relaxation timescale for turbulence distortion and an equation for effective distortion strain such as (3.3) or (4.1) also have applications to other types of flows such as uniform straining flow. In experiments on uniform straining the condition of a true rapid distortion is rarely met, with the distortion process taking place over a time comparable to the energy decay timescales. A more appropriate comparison with rapid-distortion theory would be in terms of an effective strain.

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Appendix

Under the assumptions of rapid distortion the equations for momentum fluctuations in a uniform mean shear flow $\beta(t)\, x_3\, \delta_{i1}$ are linearized:

$$\frac{\partial u_i}{\partial t} + \beta(t)\, x_3 \frac{\partial u_i}{\partial x_1} + u_3 \beta(t)\, \delta_{i1} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \tag{A 1}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{A 2}$$

The instantaneous fluctuating velocity field $u_i(\mathbf{x}, t)$ can be written as a Fourier integral (in the sense of generalized functions)

$$u_i(\mathbf{x}, t) = \int d^3\mathbf{k}\, a_i(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A 3}$$

and substituted into the above equations. The fluctuating pressure $p(\mathbf{x}, t)$ similarly has a Fourier transform $\Pi(\mathbf{k}, t)$ which because of continuity satisfies

$$\Pi(\mathbf{k}, t) = \frac{2i\rho\beta k_1 a_3}{k^2}, \qquad \text{(A 4)}$$

ρ being the fluid density. Once pressure is eliminated there are three linear equations to solve for the a_i terms. The solution is conveniently written in the form

$$a_i(\mathbf{k}, t) = A_{ij}(\mathbf{m}(\mathbf{k}, t), \alpha(t))\, a_j(\mathbf{m}(\mathbf{k}, t), 0); \qquad \text{(A 5)}$$

where the matrix $\mathbf{A}$ has components

$$A_{11} = A_{22} = 0, \qquad \text{(A 6a)}$$

$$A_{33} = \frac{m^2}{k^2} = \frac{m^2}{m^2 - 2\alpha m_1 m_3 + \alpha^2 m_1^2}, \qquad \text{(A 6b)}$$

$$A_{13} = \frac{m^2}{m_1^2 + m_2^2}\left(-\frac{m_2^2 P}{m^2} + \frac{m_1^2 Q}{m^2}\right), \qquad \text{(A 6c)}$$

$$A_{23} = \frac{m_1 m_2}{m_1^2 + m_2^2}(P + Q), \qquad \text{(A 6d)}$$

$$A_{12} = A_{21} = A_{31} = A_{32} = 0. \qquad \text{(A 6e)}$$

The functions $P(\mathbf{m}, \alpha)$ and $Q(\mathbf{m}, \alpha)$ are

$$P = \frac{m^2}{m_1(m_1^2 + m_2^2)^{\frac{1}{2}}}\left\{\arctan\frac{m_3}{(m_1^2 + m_2^2)^{\frac{1}{2}}} - \arctan\frac{m_3 - \alpha m_1}{(m_1^2 + m_2^2)^{\frac{1}{2}}}\right\}, \qquad \text{(A 7)}$$

$$Q = \frac{\alpha(m^2 - 2m_3^2 + \alpha m_1 m_3)}{m_1^2\alpha^2 - 2\alpha m_1 m_3 + m^2}. \qquad \text{(A 8)}$$

The vector $\mathbf{m}$ is the wavenumber in a frame of reference moving with mean shear flow, and is related to the wavenumber $\mathbf{k}$ in fixed coordinates by

$$(m_1, m_2, m_3) = (k_1, k_2, k_3 + k_1\alpha(t)). \qquad \text{(A 9)}$$

The turbulence distortion is characterized by the strain parameter

$$\alpha(t) = \int_0^t \frac{\partial U_1}{\partial x_3} dt' = \int_0^t \beta(t')\, dt', \qquad \text{(A 10)}$$

which is determined by the time history of the mean shear.

These results for the individual Fourier components are combined to give velocity correlations in terms of an initial spectrum tensor Φ_{ij}. If at $t = 0$ the turbulence is homogeneous, the spectrum tensor

$$\Phi_{ij}(\mathbf{m}) = (1/2\pi)^3 \int \overline{u_i(\mathbf{x}, 0)\, u_j(\mathbf{x} + \mathbf{r}, 0)}\, e^{-i\mathbf{m}\cdot\mathbf{r}}\, d^3\mathbf{r} \qquad \text{(A 11)}$$

is formally related to the Fourier modes by

$$\delta(\mathbf{m} + \mathbf{m}')\, \Phi_{ij}(\mathbf{m}) = \overline{a_i(\mathbf{m}', 0)\, a_j(\mathbf{m}, 0)}. \qquad \text{(A 12)}$$

The velocity moments $\overline{u_i u_j}$ at subsequent times are

$$\overline{u_i u_j}(t) = \int d^3\mathbf{k} \int d^3\mathbf{k}'\, \overline{a_i(\mathbf{k}, t)\, a_j(\mathbf{k}', t)}\, e^{i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')}, \qquad \text{(A 13)}$$

which from (A 5) and (A 12) are

$$\overline{u_i u_j}(t) = \int d^3\mathbf{m}\, \{A_{ip}(\mathbf{m}, \alpha(t))\, A_{jq}(\mathbf{m}, \alpha(t))\, \Phi_{pq}(\mathbf{m})\}. \qquad \text{(A 14)}$$

Similarly the pressure–strain-rate correlations can be evaluated from (A 4):

$$\sigma_{ij} = \overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}$$
$$= \int d^3\mathbf{k} \int d^3\mathbf{k}' \langle [2ik_1\beta(t)\, a_3(\mathbf{k},t)/k^2]\, [ik'_j\, a_i(\mathbf{k}',t) + ik'_i\, a_j(\mathbf{k}',t)] \rangle\, e^{i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')} \quad \text{(A 15)}$$

The terms again can be simplified using (A 5) and (A 12)

$$\sigma_{ij}(t) = \int d^3\mathbf{m}\, 2\beta(t) \left[\frac{k_1 k_j}{k^2} A_{3p}(\mathbf{m},\alpha)\, A_{iq}(\mathbf{m},\alpha) + \frac{k_1 k_i}{k^2} A_{3p}(\mathbf{m},\alpha)\, A_{jq}(\mathbf{m},\alpha)\right] \Phi_{pq}(\mathbf{m}). \quad \text{(A 16)}$$

The wavevectors $\mathbf{m}$ and $\mathbf{k}$ are related as in (A 9).

In deriving the above results the fact was used that the matrix $\mathbf{A}$ is unaltered if $\mathbf{m}$ is replaced by $-\mathbf{m}$. Furthermore, the matrix components $A_{ij}(\mathbf{m}, \alpha)$ depend only on the direction cosines of $\mathbf{m}$ and not on the magnitude m. This simplifies the evaluation, as generally the dependence on m can be separated and treated analytically, leaving the integration of (A 14) and (A 16) over spatial shells in wavenumber space to be done numerically. Other results for second-order correlations are derived in a similar manner.

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